Escape through an unstable limit cycle driven by multiplicative colored non-Gaussian and additive white Gaussian noises

Bidhan Chandra Bag[1,*] and Chin-Kun Hu[1,2,†]
[1]*Institute of Physics, Academia Sinica, Taipei 11529, Taiwan*
[2]*Center for Nonlinear and Complex Systems and Department of Physics, Chung-Yuan Christian University, Chungli 320, Taiwan*
(Received 16 October 2006; revised manuscript received 30 January 2007; published 9 April 2007)

In a previous paper [Bag and Hu, Phys. Rev. E **73**, 061107 (2006)], we studied the mean lifetime (MLT) for the escape of a Brownian particle through an unstable limit cycle driven by multiplicative colored Gaussian and additive Gaussian white noises and found resonant activation (RA) behavior. In the present paper we switch from Gaussian to non-Gaussian multiplicative colored noise. We find that in the RA phenomenon, the minimum appears at a smaller noise correlation time (τ) for non-Gaussian noises compared to Gaussian noises in the plot of MLT vs τ for a fixed noise variance; the same plot for a given noise strength increases linearly and the increasing rate is smaller for non-Gaussian noises than for the Gaussian noises; the plot of logarithm of inverse of MLT vs inverse of the strength of additive noise is Arrhenius-like for Gaussian colored noise and it becomes similar to the quantum-Kramers rate if the multiplicative noise is non-Gaussian.

DOI: 10.1103/PhysRevE.75.042101 PACS number(s): 05.40.Jc, 05.45.−a, 05.70.Ln, 05.20.−y

In recent decades, there have been many studies on the enhancement of small periodic signals by noises in nonlinear systems [1–9]. A prototypical effect of this kind, called resonant activation (RA) [2], concerns the resonance effect in the escape rate of a particle over a fluctuating barrier in a bistable potential. The phenomenon has triggered a lot of studies [3–9], including theoretical works on kinetic models for chemical reactions [3,5] and related experiments [9]. In [10], we studied the mean lifetime (MLT) [11] for the escape of a Brownian particle through an unstable limit cycle (instead of the saddle point considered in papers mentioned above) driven by multiplicative colored Gaussian and additive Gaussian white noises and found resonant activation (RA) behavior. Traditionally RA appears due to fluctuations in a nonlinear potential, our paper [10] shows that the RA phenomenon appears even in the linear potential.

Experimental data indicate that noises in biological processes have a non-Gaussian character. Examples include current through voltage-sensitive ion channels in a cell membrane, signals from the sensory systems of rat skin [12], and noise sources in different biology systems [13,14]. It is observed that biological transport works in the presence of correlated random noise of biological origin, such as the hydrolysis mechanism of adenosine 5-triphosphate (ATP) [15]. Recently, Fuentes *et al.* [16] have shown that the stochastic resonance can be enhanced when the subsystem departs from Gaussian behavior and the system shows marked 'robustness' against noise tuning, i.e., the signal-to-noise ratio curve can flatten when departing from Gaussian behavior, implying that the system does not require fine tuning of the noise intensity in order to maximize its response to a weak external signal. This theoretical finding was verified experimentally by Castro *et al.* [17]. Furthermore, non-Gaussian noise of third order has been shown to be useful in some autocatalytic reactions [18].

*On leave from Department of Chemistry, Visva-Bharati, Santiniketan, India. Electronic address: pcbcb@rediffmail.com

†Author to whom correspondence should be addressed. Electronic address: huck@phys.sinica.edu.tw

Considering the importance of non-Gaussian noise, here we examine how the RA behavior changes if one switches from Gaussian to non-Gaussian colored multiplicative noise. We show that the plot of $\ln(1/\langle T\rangle)$ vs inverse of strength of additive noise is nonlinear if the multiplicative noise is non-Gaussian, but it becomes linear for the Gaussian multiplicative noise. The latter is similar to the well known plot of Arrhenius law and the former resembles that of the quantum escape kinetics [19].

Model and acting noises. We consider the following Langevin equation of motion [10]:

$$\dot{v} = -aq + b(v^2-1)v + q\eta(t) + \zeta(t), \tag{1}$$

where q and $v \equiv \dot{q}$ represent, respectively, the coordinate and the velocity of the Brownian particle. On the right hand side, the first term is due to the force derived from harmonic potential, the second term captures essential features of negative and positive feedback of biological systems, $\eta(t)$ is a colored noise and may be either Gaussian or non-Gaussian; $\zeta(t)$ is a Gaussian white noise and is characterized by $\langle\zeta(t)\rangle=0$ and $\langle\zeta(t)\zeta(t')\rangle=2D'\delta(t-t')$ with D' [11] being the noise strength.

It is very relevant to understand the transition between attractors through an unstable limit cycle. The transition through an unstable limit cycle occurs in the presence of noise in the context of the autocatalytic biochemical system [20], the biological oscillations [21], and the chemical reactions constrained to happen far from equilibrium [22]. Unstable limit cycles often appear [23] in a multidimensional biological system to separate (i) a stable fixed point and limit cycle, (ii) two stable limit cycles, (iii) two stable fixed points, etc. To consider the effect of biological environments, we include multiplicative non-Gaussian noise in the model, which can make the system far from equilibrium. We also include additive noise to take care of thermal fluctuations.

The colored noise η can be generated as the solution of the following stochastic differential equation [24]:

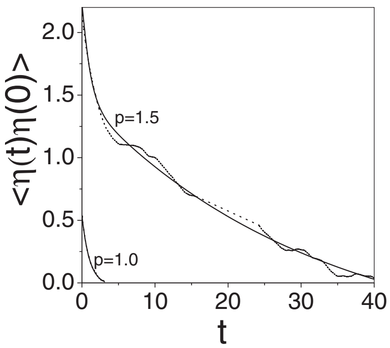

FIG. 1. Two-time correlation function vs t of both Gaussian and non-Gaussian noises for parameters $\tau=1.0$ and $D=0.5$.

$$\dot{\eta}=-\frac{1}{\tau}\frac{d}{d\eta}U_p(\eta)+\frac{\sqrt{D}}{\tau}\zeta_1(t). \tag{2}$$

Here D [11] and τ are, respectively, the noise intensity and the correlation time, $\zeta_1(t)$ is a standard Gaussian noise of zero mean and its two-time correlation is given by $\langle\zeta_1(t)\zeta_1(t')\rangle=2\delta(t-t')$ and $U_p(\eta)=[D/\tau(p-1)]\ln[1+\alpha(p-1)\eta^2/2]$, where $\alpha=\tau/D$. The form for noise $\eta(t)$ allows us to control the departure from the Gaussian behavior easily by changing a single parameter p. For $p=1$, Eq. (2) becomes $\dot{\eta}=-\frac{\eta}{\tau}+\frac{\sqrt{D}}{\tau}\zeta_1(t)$, which is a well known time evolution equation of the Ornstein-Uhlenbeck noise process for which the correlation function $\langle\eta(t)\eta(0)\rangle$ decays exponentially, $\langle\eta(t)\eta(0)\rangle\sim\frac{D}{\tau}e^{-t/\tau}$. Thus τ is the correlation time of the Ornstein-Uhlenbeck noise. To have an idea about the correlation time for non-Gaussian noise we have plotted the two-time correlation function vs time in Fig. 1 based on numerical simulation. The curve for non-Gaussian noise (dotted curve) is fitted well by the bi-exponential decaying function (solid curve) with correlation times 31 and 1, respectively, for $p=1.5$. Figure 1 shows that the effective correlation time and the noise strength for $p>1$ is larger than those for $p=1.0$.

The stationary probability distribution of η is [25]

$$P(\eta)=\frac{1}{Z_p}\left[1+\alpha(p-1)\frac{\eta^2}{2}\right]^{-1/(p-1)}, \tag{3}$$

where $Z_p=\sqrt{\frac{\pi}{\alpha(p-1)}}\frac{\Gamma[1/(p-1)-1/2]}{\Gamma[1/(p-1)]}$ is the normalization factor with Γ being the Gamma function. This distribution can be normalized only for $p<3$. Since $P(\eta)$ is an even function of η, the first moment $\langle\eta\rangle$ is zero, and the second moment (noise variance), given by

$$\langle\eta_p^2\rangle=\frac{2D}{\tau(5-3p)}, \tag{4}$$

is finite only for $p<5/3$. Furthermore, for $p<1$, the distribution has a cutoff and it is defined only for $|\eta|<\eta_c\equiv\sqrt{\frac{2D}{\tau(1-p)}}$.

When $p\to1$, the term in the square brackets of Eq. (3) becomes $\exp[\alpha(p-1)\eta^2/2]$, Eq. (3) becomes $P(\eta)=\frac{1}{Z_1}\exp(-\alpha\eta^2/2)$ with $Z_1=\sqrt{\pi/\alpha}$, and η becomes a Gaussian colored noise. Equation (4) shows that for a given D and τ the variance of the non-Gaussian is higher than that of the Gaussian noise for $p>1$, i.e., $\langle\eta_p^2\rangle>\langle\eta^2\rangle$.

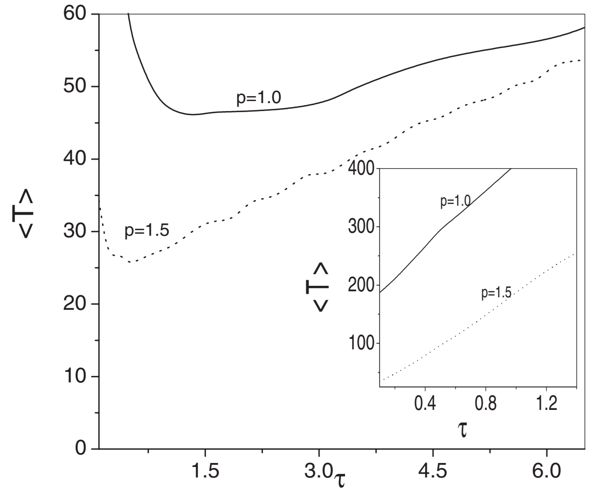

FIG. 2. Mean lifetime $\langle T\rangle$ vs the noise correlation time τ of the multiplicative colored noise with fixed noise variance for the same parameter set $a=b=1.0$, $C=0.5$, and $D'=0.05$. The inset shows $\langle T\rangle$ vs τ with fixed noise strength for $a=b=1.0$, $D=0.1$, and $D'=0.05$.

Method and results. It is difficult to deal with the problem analytically because of the finite correlation time of the multiplicative noise and the nonlinearity in velocity and η in Eqs. (1) and (2), respectively. Therefore we have solved the differential equations (1) and (2) simultaneously using Heun's method, stochastic variant of the Euler method which reduces to the second order Runge-Kutta method in the absence of noise [26]. We define the lifetime or exit time (T) as the time required for the particle to go from the origin of phase space (0,0) to the point where $v=2$ or $v=-2$ for the first time. The average T over many realizations (say, 5000) gives the mean lifetime (MLT) or exit time ($\langle T\rangle$) [10,11]. We define the lifetime using this boundary value of v because at this value the phase point is definitely out of the basin of attraction. When the parameter b in Eq. (1) increases, the boundary value of velocity decreases since even at low value of v positive feedback becomes sufficient to escape the basin of attraction. Similarly, the boundary value of v decreases with increase of its nonlinearity in the positive feedback term.

Similarly, one can define MLT using the boundary value of coordinate q. The boundary value for q decreases with increase of system parameter a: when a increases, the frequency of the harmonic oscillator increases and the phase point is strongly localized near the origin. When a tends to 0, the particle can easily escape to large v and q when $|v|>1$.

To investigate how the RA phenomenon is affected when one switches from Gaussian to non-Gaussian multiplicative color noise, we have calculated MLT at different noise correlation time τ and plotted the results in Fig. 2. In this figure the noise variance C is kept fixed and the noise strength D increases linearly with τ as $D=C\tau$, which is obtained from the intuition that the nonequilibrium potential [27] in the present model might have a similar role as that of the nonlinear potential [7] in the ordinary barrier crossing dynamics.

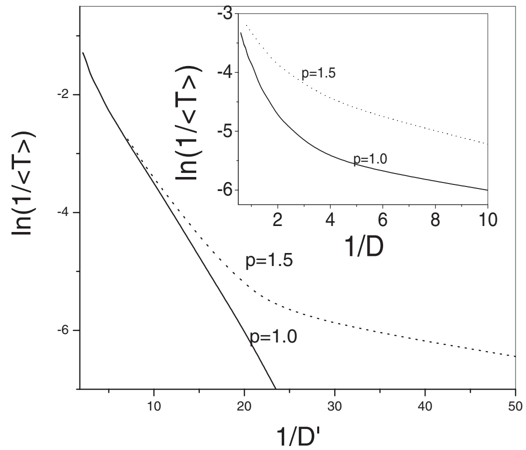

FIG. 3. $\ln(1/\langle T\rangle)$ vs $1/D'$, inverse of the strength of additive white noise for $a=b=\tau=1.0$, $D=0.1$. The inset shows $\ln(1/\langle T\rangle)$ vs $(1/D)$ for $a=b=\tau=1.0$ and $D'=0.05$.

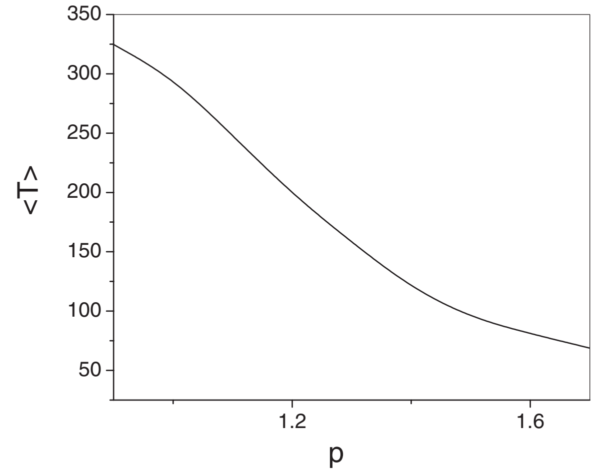

FIG. 4. Mean lifetime $\langle T\rangle$ vs the non-Gaussian parameter p for $a=b=1.0$, $\tau=0.5$, $D=0.1$, and $D'=0.05$.

It has been used by several authors [6,10,28,29] for the study of resonant activation. Substituting $D=C\tau$ in Eq. (4) one can easily check that for a given value of C and τ the variance of the non-Gaussian noise is greater than that of the Gaussian noise for $p>1$ and does not change with increase of τ. Figure 1 shows that $\langle T\rangle$ first decreases, followed by an increase after passing through a minimum; the dotted and the solid curves are for non-Gaussian and Gaussian noises, respectively. This convention will be followed for other figures.

The decrease of $\langle T\rangle$ for the increase of noise correlation time might be due to the decrease of effective barrier height with the increase of the multiplicative noise strength by the relation $D=C\tau$ analogous to the escape through the saddle point [7]. If the noise correlation time is sufficiently large then the frequency factor of the escape rate rapidly decreases for rising τ [7]. As a result of the interplay of these two factors the mean lifetime first decreases and then increases after passing through a minimum for the increase of noise correlation time and strength by the relation $D=C\tau$. Figure 2 shows that the minimum appears for the non-Gaussian noise at a smaller correlation time compared to the Gaussian noise. Because of higher effective noise strength for the former than the latter for a given D [see Eq. (4), also see Fig. 1 where we plot numerical results on two-time correlation of Gaussian and non-Gaussian noises, respectively] the effective barrier height decreases at faster rate with increase of τ for the non-Gaussian noise than the Gaussian noise and therefore the minimum first appears for the former. Due to greater effective noise correlation time for the $p>1$ case (see Fig. 1) the rate of increase of the MLT is higher for $p>1$ than the case $p=1$.

Now we investigate how $\langle T\rangle$ changes with τ when noise strength is kept fixed. In the inset of Fig. 2, we plot $\langle T\rangle$ vs τ for a given value of D for both the Gaussian and non-Gaussian noises. Although the MLT first increases nonlinearly then reaches a limiting value [29] for nonlinear potential, in the present model it rises linearly with τ in both cases. The rate of increase of $\langle T\rangle$ for the non-Gaussian noise is slower compared to the Gaussian one because of greater effective noise strength for the former than for the latter.

In the next step we plot the logarithm of $1/\langle T\rangle$ vs $1/D'$ in Fig. 3 which shows that the plot is linear for multiplicative Gaussian noise (it is well known for the escape of a particle through a saddle point) and the plot for non-Gaussian noise is similar to what was observed in the case of the quantum-Kramers rate [19]. Because of the higher effective noise strength of non-Gaussian noise than Gaussian noise we find a finite barrier crossing rate even at very low noise strength of the additive white noise. Now we study the behavior of this plot for the variation of strength of multiplicative noise D keeping fixed the strength of additive noise D'. In the inset of Fig. 3, we have plotted logarithm of $1/\langle T\rangle$ vs $1/D$. It exhibits that the plot is exponentially decaying at a large value of the multiplicative noise strength D and becomes linear at low noise strength. Thus the multiplicative noise strength affects both the frequency and exponential factors of barrier crossing rate expression when the strength is large and the frequency factor becomes independent of it in the weak noise limit. The decay rate at large D is faster for the Gaussian noise than the non-Gaussian one due to greater effective noise strength of the latter than the former.

Finally, we plot the mean lifetime $\langle T\rangle$ as a function of non-Gaussian parameter p in Fig. 4, which shows that $\langle T\rangle$ decreases as p increases and at large p it trends to a limiting value. This is due to the following fact. Both the effective noise strength (4) and correlation time become larger as p increases. The former enhances the rate but the latter decreases it (the barrier height increases and the frequency factor decreases with increases of noise correlation time). At $p\to 1$ the noise strength dominates over the noise correlation and at large p they balance each other.

We have applied the color noise generated by Eq. (2) to the Langevin equation for a particle in the overdamped limit and found that the particle has anomalous diffusion [30]. It is of interest to find other applications of Eq. (1) and the noise generated by Eq. (2).

This work was supported by the National Science Council in Taiwan under Grant No. NSC 95-2112-M 001-008 and the National Center for Theoretical Sciences in Taiwan.

[1] For reviews, see F. Julicher, A Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997); L. Gamaitoni, P. Hänggi, P. Jung, and F. Marchesoni, *ibid.* **70**, 223 (1998).
[2] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992); J. Maddox, Nature (London) **359**, 771 (1992).
[3] M. Bier and R. D. Astumian, Phys. Rev. Lett. **71**, 1649 (1993); U. Zürcher and C. R. Doering, Phys. Rev. E **47**, 3862 (1993); C. Van den Broeck, *ibid.* **47**, 4579 (1994); J. J. Brey and J. Casado-Pascual, *ibid.* **50**, 116 (1994); O. Flomenbom and J. Klafter, *ibid.* **69**, 051109 (2004).
[4] T. D. Shepherd and R. Hernandez, J. Chem. Phys. **115**, 2430 (2001); A. M. Berezhkovskii, A. Szabo, G. H. Weiss, and H. Zhou, *ibid.* **111**, 9952 (1999).
[5] M. Boguñá, J. M. Porrá, J. Masoliver, and K. Lindenberg, Phys. Rev. E **57**, 3990 (1998).
[6] M. Marchi F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, S. Santucci, Phys. Rev. E **54**, 3479 (1996).
[7] P. Hänggi, Chem. Phys. **180**, 157 (1994).
[8] R. F. Fox, Phys. Rev. A **34**, R4525 (1986); Y. Jia and J. R. Li, Phys. Rev. E **53**, 5786 (1996); X. Luo and S. Zhu, *ibid.* **67**, 021104 (2003).
[9] R. N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **84**, 3025 (2000).
[10] B. C. Bag and C. K. Hu, Phys. Rev. E **73**, 061107 (2006).
[11] MLT, D and D' in the present paper correspond to MFPT, D_0, and D, respectively, in Ref. [10].
[12] S. M. Bezrukov and I. Vodyanoy, Nature (London) **378**, 362 (1995); D. Nozaki D. J. Mar, P. Grigg, J. J. Collins, Phys. Rev. Lett. **82**, 2402 (1999).
[13] A. Manwani and C. Koch, Neural Comput. **11**, 1797 (1999).
[14] A. Manwani, Ph.D. thesis, Caltech, 2000.
[15] S. Leibler, Nature (London) **370**, 412 (1994).
[16] M. A. Fuentes, R. Toral, and H. S. Wio, Physica A **295**, 114 (2001).
[17] F. J. Castro, M. N. Kuperman, M. Fuentes, and H. S. Wio, Phys. Rev. E **64**, 051105 (2001).
[18] C. W. Gradiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Processes* (Springer-Verlag, New York, 1985).
[19] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[20] F. Moran and A. Goldbeter, Biophys. Chem. **20**, 149 (1984).
[21] J. D. Murray, *Mathmatical Biology* (Springer-Verlag, New York, 1991).
[22] B. Gaveau E. Gudowska-Nowak, R. Kapral, M. Moreau, Phys. Rev. A **46**, 825 (1992); I. L'Heureux and R. Kapral, Phys. Lett. A **136**, 472 (1989); P. Rehmus, W. Vance, and J. Ross, J. Chem. Phys. **80**, 3373 (1984).
[23] Z. Z. Sun, K. L. Chen, S. Yin, H. T. He, J. N. Wang, Y. Q. Wang, and X. R. Wang, New J. Phys. **6**, 148 (2004); J. Liu and J. W. Crawford, IMA J. Math. Appl. Med. Biol. **15**, 339 (1998); V. I. Nekorkin and V. A. Makarov, Phys. Rev. Lett. **74**, 4819 (1995); A. Hastings, J. Math. Biol. **11**, 51 (1981); V. Krivan and J. Eisner, Theor Popul. Biol. **63**, 269 (2003).
[24] L. Borland, Phys. Lett. A **245**, 67 (1998).
[25] M. A. Fuentes, H. S. Wio, and R. Toral, Physica A **303**, 91 (2002).
[26] R. Toral, in *Computational Physics*, Lecture Notes in Physics Vol. 448, edited by P. Garrido and J. Marro (Springer-Verlag, Berlin, 1995).
[27] T. Naeh, M. M. Klosek, B. J. Matkowsky, and Z. Schuss, SIAM J. Appl. Math. **50**, 595 (1990); R. Graham and T. Tél, Phys. Rev. Lett. **52**, 9 (1984); Phys. Rev. A **31**, 1109 (1985); M. V. Day, in *Diffusion Processes and Related Problems in Analysis*, edited by M. Pinsky (Birkhäuser, 1990).
[28] J. Iwaniszewski, Phys. Rev. E **54**, 3173 (1996); P. K. Ghosh, D. Barik, B. C. Bag, and D. S. Ray, J. Chem. Phys. **123**, 224104 (2005).
[29] P. Reimann *et al.*, Chem. Phys. **235**, 11 (1998).
[30] B. C. Bag and C.-K. Hu (unpublished); for anomalous diffusion, see, e.g., X.-J. Wang and C.-K. Hu, Phys. Rev. E **48**, 728 (1993).